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$$Q = -\frac{\beta\gamma d \log a + \gamma a d \log b + a\beta d \log c}{a\beta + a\gamma + \beta\gamma}.$$

Therefore by (3)

$$dA = \frac{\beta(d \log a - d \log c) + \gamma(d \log a - d \log b)}{a\beta + a\gamma + \beta\gamma},$$

$$dB = \frac{\gamma(d \log b - d \log a) + a(d \log b - d \log c)}{a\beta + a\gamma + \beta\gamma},$$

$$dC = \frac{a(d \log c - d \log b) + \beta(d \log c - d \log a)}{a\beta + a\gamma + \beta\gamma}.$$

SOLUTION BY PROF. ASAPH HALL, NAVAL OBSERVATORY, WASH., D. C.

Projecting the sides  $b$  and  $c$  of a plane triangle on the side  $a$  we have

$$a = b \cos C + c \cos B;$$

and in a similar manner we find two more equations of this kind. Differentiating these, considering all the parts variable and noticing the condition

$$A + B + C = \pi = \text{constant},$$

we have the three symmetrical differential equations of a plane triangle,

$$da = \cos C.db + \cos B.dc + c \sin B.dA,$$

$$db = \cos A.dc + \cos C.da + a \sin C.dB,$$

$$dc = \cos B.da + \cos A.db + b \sin A.dC.$$

The quantities  $\cos C.db$  and  $\cos B.dc$  are the increments of the sides  $b$  and  $c$  projected on the side  $a$ ; and the sum of these applied to  $da$  gives the total increment of the side  $a$ . Also  $c \sin B$  is the perpendicular from the angle  $A$  on the opposite side, and the total increment of  $a$  divided by  $c \sin B$  gives  $dA$ . For seconds of arc we must multiply this ratio by 206264.8, the number of seconds in radius. Since

$$da = \frac{a.da}{a} = a.d \log a,$$

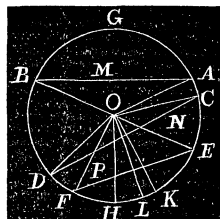
we may use the differentials themselves or  $d \log a$ , &c. One should notice the similarity of these differential equations to those of spherical trigonometry.

SOLUTION OF PART (2), PROB. 344, BY PROF. SEITZ (SEE P. 101).—Let  $P$  be the third random point, and  $EF$  the random chord, through it.

Draw the radius  $OL$  perpendicular to  $EF$ . Let  $p_0, p_1, p_2, p_3$  be the respective probabilities that  $AB, CD, EF$  will intersect in 0, 1, 2, 3 points.

Let  $EP = z, EF = z', \angle EOL = \phi$ , and  $\angle KOL = \rho$ . Then  $z' = 2r \times \sin \phi$ ; an element of the circle at  $P$  is  $r \sin \phi d\phi dz$ , and for an elemental change in the direction of  $EF$  we have  $d\rho$ .

For non-intersection, the chord  $CD$  being between  $AB$  and  $EF$ , the limits of  $\theta$  are 0 and  $\pi$ ; of  $\varphi$ , 0 and  $\theta$ ; of  $\psi$ , 0 and  $\varphi$ ; of  $\mu$ ,  $\varphi - \theta$  and  $\theta - \varphi$ ; and of  $\rho$ ,  $\psi - \varphi$  and  $\varphi - \psi$ . The result of these integrations must be multiplied by 6 to allow for the interchange of the chords.



For non-intersection, the chord  $EF$  having its extremities in the arc  $BD$ , the limits of  $\theta$  are 0 and  $\pi$ ; of  $\varphi$ , 0 and  $\theta$ ; of  $\psi$ , 0 and  $\theta - \varphi$ ; of  $\mu$ ,  $2\psi - \theta + \varphi (= \alpha)$  and  $\theta - \varphi$ ; and of  $\rho$ ,  $\varphi + \psi$  and  $\theta + \mu - \psi (= \beta)$ . This result must be multiplied by 4 to allow for the positions of  $EF$ , in which its extremities are in the arc  $AC$ , and for the positions of  $CD$  and  $EF$  in which their extremities are in the arc  $AGB$ .

The limits of  $\omega$  are 0 and  $\pi$ ; of  $x$ , 0 and  $x'$ ; of  $y$ , 0 and  $y'$ ; and of  $z$ , 0 and  $z'$ . Hence, since the whole number of ways the three chords can be drawn is  $(\pi r^2)^3 \cdot \pi^3 = \pi^6 r^6$ , we have

$$p_0 = \frac{6}{\pi^6 r^6} \int_0^\pi \int_0^\theta \int_0^\phi \int_{\phi-\theta}^{\theta-\phi} \int_{\psi-\phi}^{\phi-\psi} \int_0^\pi \int_0^{x'} \int_0^{y'} \int_0^{z'} r^3 \sin \theta \sin \varphi \sin \psi d\theta d\varphi d\psi d\mu \\ \times d\rho d\omega dx dy dz \\ + \frac{4}{\pi^6 r^6} \int_0^\pi \int_0^\theta \int_0^{\theta-\phi} \int_a^{\theta-\phi} \int_{\phi+\psi}^\beta \int_0^\pi \int_0^{x'} \int_0^{y'} \int_0^{z'} r^3 \sin \theta \sin \varphi \sin \psi d\theta d\varphi d\psi \\ \times d\mu d\rho d\omega dx dy dz \\ = \frac{1}{3} - \frac{5}{\pi^2} + \frac{105}{4\pi^4}.$$

We will next find the probability that the chords  $CD$  and  $EF$  will both intersect  $AB$ . By reference to the first part of the solution we readily see that the probability that a third chord will also intersect  $AB$  is

$$P = \frac{16}{\pi^5} \int_0^{\frac{1}{2}\pi} \left\{ 2 \int_0^\theta \int_{\theta-\phi}^{\theta+\phi} \sin^2 \varphi d\varphi d\mu + 2 \int_\theta^{\frac{1}{2}\pi} \int_{\phi-\theta}^{\phi+\theta} \sin^2 \varphi d\varphi d\mu \right\}^2 \sin^2 \theta d\theta \\ = \frac{2}{15} + \frac{1}{\pi^2} + \frac{49}{4\pi^4}.$$

Now  $P$  is evidently equal to the probability that the chords  $CD$  and  $EF$  will both intersect  $AB$ , and not each other, plus  $p_3$ ;  $\therefore \frac{1}{3}p_2 + p_3 = P$ . (1)

The probability  $p$ , that  $AB$  and  $CD$  will intersect, is equal to the probability that they will intersect each other, and not  $EF$ , plus the probability that  $CD$  and  $EF$  will both intersect  $AB$ , and not each other, plus the prob. that  $AB$  and  $EF$  will both intersect  $CD$ , and not each other, plus  $p_3$ ; hence

$$\frac{1}{3}p_1 + \frac{2}{3}p_2 + p_3 = p. \quad (2)$$

We also have

$$p_1 + p_2 + p_3 = 1 - p_0, \quad (3)$$

From (1), (2) and (3), knowing the values of  $P$ ,  $p$  and  $p_0$ , we find

$$p_1 = \frac{2}{5} + \frac{3}{\pi^2} - \frac{42}{\pi^4}, \quad p_2 = \frac{1}{5} + \frac{3}{2\pi^2} + \frac{21}{4\pi^4}, \quad p_3 = \frac{1}{15} + \frac{1}{2\pi^2} + \frac{21}{2\pi^4}.$$